The unicity of real Picard–Vessiot fields

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Abstract

Using Deligne's work on Tannakian categories, the unicity of real Picard-Vessiot fields for differential modules over a real differential field is derived. The inverse problem for real forms of a semi-simple group is treated. Some examples illustrate the relations between differential modules, Picard–Vessiot fields and real forms of a group.

1 Introduction

K denotes a real differential field with field of constants k. We suppose that $k \neq K$ and that k is a real closed field. Let M denote a differential module over K of dimension d, represented by a matrix differential equation $y' = Ay$ where A is a $d \times d$-matrix with entries in K. A *Picard–Vessiot field* L for M/K is a field extension of K such that:
(a) L is equipped with a differentiation extending the one of K,
(b) M has a full space of solutions over L, i.e., there exists an invertible $d \times d$-matrix F (called a fundamental matrix) with entries in L satisfying $F' = AF$,
(c) L is (as a field) generated over K by the entries of F,
(d) the field of constants of L is again k.

A *real Picard–Vessiot field* L for M/K is a Picard–Vessiot field which is also a real field. In [CHS1] and [CHS2] the existence of a real Picard–Vessiot field is proved using results of Kolchin.

The main result of this paper is:

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Theorem 1.1 *Let L_1, L_2 denote two real Picard–Vessiot extensions for M/K. Suppose that L_1 and L_2 have total orderings which induce the same total ordering on K. Then there exists a K-linear isomorphism $\phi : L_1 \to L_2$ of differential fields.*

Remarks 1.2 Suppose that ϕ exists. Choose a total ordering of L_1 and define the total ordering of L_2 to be induced by ϕ. Then L_1 and L_2 induce the same total ordering on K. Therefore the condition of the Theorem 1.1 is necessary.

If K happens to be real closed, then the assumption in the theorem is superfluous since K has a unique total ordering. On the other hand, consider the example $K = k(z)$ with differentiation $' = \frac{d}{dz}$ and the equation $y' = \frac{1}{2z}y$. Let $L_1 = K(t_1)$ with $t_1^2 = z$ and $L_2 = K(t_2)$ with $t_2^2 = -z$. Both fields are real Picard–Vessiot fields for this equation. They are not isomorphic as differential field extensions of K, since z is positive for any total ordering of L_1 and z is negative for any total ordering of L_2. $\square$

The proof of Theorem 1.1 uses Tannakian categories as presented in [DM] and P. Deligne's fundamental paper [De]. We adopt much of the notation of [De]. Let $< M >_\otimes$ denote the Tannakian category generated by the differential module M. The forgetful functor $\rho :< M >_\otimes \to vect(K)$ associates to any differential module $N \in < M >_\otimes$ the finite dimensional K-vector space N. Let $\omega :< M >_\otimes \to vect(k)$ be a fibre functor with values in the category $vect(k)$ of the finite dimensional vector spaces over k.

Now we recall some results of [De], §9. The functor $\underline{Aut}^\otimes(\omega)$ is represented by a linear algebraic group G over k. By Proposition 9.3, the functor $\underline{Isom}_K^\otimes(K \otimes \omega, \rho)$ is represented by a torsor P over $G_K := K \times_k G$. This torsor is affine, irreducible and its coordinate ring $O(P)$ has a natural differentiation extending the differentiation of K. Moreover, the field of fractions $K(P)$ of $O(P)$ is a Picard–Vessiot field for M/K.

On the other hand, let L be a Picard–Vessiot field for M/K. Define the fibre functor $\omega_L :< M >_\otimes \to vect(k)$ by $\omega_L(N) = \ker(\partial : L \otimes_K N \to L \otimes_K N)$. Then ω_L produces a Picard–Vessiot field L' which is isomorphic to L as differential field extension of K. The conclusion is:

Proposition 1.3 ([De],§9) *The above constructions yield a bijection between the (isomorphy classes of) fibre functors $\omega :< M >_\otimes \to vect(k)$ and the (isomorphy classes of) Picard–Vessiot fields L for M/K.*

The following result will also be useful.

Proposition 1.4 ([DM], Thm. 3.2) *Let $\omega :< M >_\otimes \to vect(k)$ be a fibre functor and $G = \underline{Aut}_k^\otimes(\omega)$.*
(a). *For any field $F \supset k$ and any fibre functor $\eta :< M >_\otimes \to vect(F)$, the functor $\underline{Isom}_F^\otimes(F \otimes \omega, \eta)$ is representable by a torsor over $G_F = F \times_k G$.*
(b). *The map $\eta \mapsto \underline{Isom}_F^\otimes(F \otimes \omega, \eta)$ is a bijection between the (isomorphy classes of) fibre functors $\eta :< M >_\otimes \to vect(F)$ and the (isomorphy classes of) G_F-torsors.*

The main ingredient in the proof of Theorem 1.1, given in §2, is:

Theorem 1.5 *Suppose that K is real closed. Let L be a Picard–Vessiot field for M/K. Then L is a real field if and only if the torsor $\underline{Isom}_K^\otimes(K \otimes \omega_L, \rho)$ is trivial.*

2 The proof of Theorem 1.1

2.1 Reduction to K is a real closed differential field

For notational convenience, the differential module M/K is represented by a scalar homogeneous linear differential equation $\mathcal{L}(y) := y^{(d)} + a_{d-1}y^{(d-1)} + \cdots + a_1 y^{(1)} + a_0 y = 0$. A Picard–Vessiot field L for $\mathcal{L}$ has the properties:
k is the field of constants of L, the *solution space* $V = \{v \in L | \ \mathcal{L}(v) = 0\}$ is a k-linear space of dimension d and L is generated over the field K by V and all the derivatives of the elements in V. One writes $L = K < V >$ for this last property.

Lemma 2.1 *Let L_1, L_2 be two real Picard–Vessiot fields for M over K. Suppose that L_1 and L_2 have total orderings extending a total ordering τ on K. Let $K^r \supset K$ be the real closure of K inducing the total ordering τ. Then:*

The fields L_1, L_2 induce Picard–Vessiot fields $\tilde{L}_1, \tilde{L}_2$ for $K^r \otimes M$ over K^r. These fields are isomorphic as differential field extensions of K^r if and only if L_1 and L_2 are isomorphic as differential field extensions of K.

Proof. Let, for $j = 1,2$, τ_j be a total ordering on L_j inducing τ on K and let L_j^r be the real closure of L_j which induces the ordering τ_j. The algebraic closure K_j of K in L_j^r is real closed. Since τ_j induces τ, there exists a K-linear isomorphism $\phi_j : K^r \to K_j$. This isomorphism is unique since the only K-linear automorphism of K^r is the identity. We will identify K_j with K^r.

Let $V_j \subset L_j$ denote the solution space of M. Then, for $j = 1,2$, the field $\tilde{L}_j := K^r < V_j > \subset L_j^r$ is a real Picard–Vessiot field for $K^r \otimes M$.

Assume the existence of a K^r-linear differential isomorphism $\psi : K^r < V_1 > \to K^r < V_2 >$. Clearly $\psi(V_1) = V_2$ and ψ induces therefore a K-linear differential isomorphism $L_1 = K < V_1 > \to L_2 = K < V_2 >$.

On the other hand, an isomorphism $\phi : L_1 \to L_2$ (of differential field extensions of K) extends to an isomorphism $\tilde{\phi} : L_1^r \to L_2^r$. Clearly $\tilde{\phi}$ maps $\tilde{L}_1$ to $\tilde{L}_2$. $\square$

2.2 Real algebras and connected linear groups

An algebra R (commutative with 1 and without zero divisors) is called *real* if $x_1, \dots, x_n \in R$ and $\sum_{j=1}^n x_j^2 = 0$ implies $x_1 = \cdots = x_n = 0$. By lack of a reference we give a proof of the following statement.

Lemma 2.2 *Let F be a real closed field and let G be a linear algebraic group over F such that $G_{F(i)} = F(i) \times_F G$ is connected. Then the coordinate ring $F[G]$ of G over F is a real algebra.*

Proof. We consider the case F is equal to $\mathbb{R}$, the field of real numbers. Consider $x_1, \dots, x_n \in \mathbb{R}[G]$ with $\sum_{j=1}^n x_j^2 = 0$. We regard $G(\mathbb{R})$ as a real analytic group. There is an exponential map $Lie(G)(\mathbb{R}) \to G(\mathbb{R})$, where $Lie(G)$ is the Lie algebra of G. Define the real analytic map $\tilde{x_j} : Lie(G)(\mathbb{R}) \stackrel{exp}{\to} G(\mathbb{R}) \stackrel{x_j}{\to} \mathbb{R}$. Now $\sum \tilde{x}_j^2 = 0$ and hence all $\tilde{x_j} = 0$. The complex analytic morphism $X_j : Lie(G)(\mathbb{C}) \stackrel{exp}{\to} G(\mathbb{C}) \stackrel{x_j}{\to} \mathbb{C}$ is the complex extension of $\tilde{x_j}$. It is zero since it is zero on the subset $Lie(G)(\mathbb{R})$ of $Lie(G)(\mathbb{C})$. The image of the complex exponential map generates the component of the identity of $G(\mathbb{C})$ and x_j is zero on this set. By assumption $G_{\mathbb{C}}$ is connected and thus $x_j = 0$ is zero for all j. Hence $\mathbb{R}[G]$ is a real algebra.

For any real field F which has an embedding in $\mathbb{R}$, one has $F[G] \subset \mathbb{R}[G]$ and $F[G]$ is a real algebra. Further, a real field k which is finitely generated over $\mathbb{Q}$ has an embedding in $\mathbb{R}$ ([Si] Proposition 3).

We consider the general case: G is a linear algebraic group defined over a real closed field F. Now G is defined over a subfield F_0 of F which is finitely generated over $\mathbb{Q}$. Then $F[G]$ is the union of the subrings $k[G]$, where k runs in the set of the subfields of F which are finitely generated over $\mathbb{Q}$ and contain F_0. Thus $F[G]$ is a real algebra since every $k[G]$ is a real algebra. $\square$

Remark. The condition that $G_{F(i)}$ is connected (or equivalently G is connected) is necessary. Indeed, consider the example of the group μ_3 over $\mathbb{R}$ with coordinate algebra $\mathbb{R}[X]/(X^3-1)$. This algebra is isomorphic to the direct sum $\mathbb{R} \oplus \mathbb{R}[X]/(X^2+X+1)$ and therefore is not real.

Corollary 2.3 ([La] Corollary 6.8) *Let G be a linear algebraic group over the real closed field F. Suppose that $G_{F(i)}$ is connected. Then the group $G(F)$ is Zariski dense in $G(F(i))$.*

Theorem 2.4 (1.5) *Suppose that K is real closed. Let L be a Picard–Vessiot field for a differential module M/K. Then L is a real field if and only if the torsor $\underline{Isom}_K^{\otimes}(K \otimes \omega_L, \rho)$ is trivial.*

Proof. $G := \underline{Aut}_k^{\otimes}(\omega_L)$ coincides with the group of the K-linear differential automorphisms of L. Let R denote the coordinate ring of the torsor $\underline{Isom}_K^{\otimes}(K \otimes \omega_L, \rho)$. Then L is the field of fractions of R.

If L is a real Picard–Vessiot field, then $R \subset L$ is a finitely generated real K-algebra. From the real Nullstellensatz and the assumption that K is real closed it follows that there exists a K-linear homomorphism $\phi : R \to K$ with $\phi(1) = 1$. The torsor $Spec(R)$ has a K-valued point and is therefore trivial.

We observe that $L(i)$ is a Picard–Vessiot field for the differential module $K(i) \otimes M$ over $K(i)$. Further $G_{k(i)}$ is the group of the $K(i)$-linear differential automorphisms of $L(i)$ and is the 'usual' differential Galois group of $K(i) \otimes M$ over $K(i)$. This group is connected since $K(i)$ is algebraically closed.

Suppose that the torsor $Spec(R)$ is trivial. Then $R \cong K \otimes_k k[G] \cong K[G]$. According to Lemma 2.2, $K[G]$ is a real K-algebra and therefore its field of fractions L is a real field. $\square$

2.3 The final step

By Lemma 2.1, we may suppose that K is real closed. Let L_1, L_2 denote two real Picard-Vessiot fields for a differential module M/K.

Write $\omega_j = \omega_{L_j} :< M >_\otimes \to vect(k)$ for the corresponding fibre functors. Let $G = \underline{Aut}_k^\otimes(\omega_1)$. Then $\underline{Isom}_k^\otimes(\omega_1, \omega_2)$ is a G-torsor over k corresponding to an element $\xi \in H^1(\{1,\sigma\}, G(k(i)))$, where $\{1,\sigma\}$ is $Gal(k(i)/k)$, represented by a 1-cocycle c with $c(1) = 1,\ c(\sigma) \in G(k(i))$ and $c(\sigma) \cdot {}^\sigma c(\sigma) = 1$.

The G_K-torsor $\underline{Isom}_K^\otimes(K \otimes \omega_1, K \otimes \omega_2)$ corresponds to an element $\eta \in H^1(\{1,\sigma\}, G(K(i)))$. This element is the image of ξ under the map, induced by the inclusion $G(k(i)) \subset G(K(i))$, from $H^1(\{1,\sigma\}, G(k(i)))$ to $H^1(\{1,\sigma\}, G(K(i)))$ (we note that $Gal(K(i)/K) = Gal(k(i)/k)$). Since L_j is real, the torsor $\underline{Isom}_K^\otimes(K \otimes \omega_j, \rho)$ is trivial for $j = 1, 2$, by Theorem 1.5. Thus there exists isomorphisms $\alpha_j : K \otimes \omega_j \to \rho$ for $j = 1, 2$. The isomorphism $\alpha_2^{-1} \circ \alpha_1 : K \otimes \omega_1 \to K \otimes \omega_2$ implies that η is trivial. In particular, there is an element $h \in G(K(i)))$ such that $c(\sigma) = h^{-1}\sigma(h)$.

There exists a finitely generated k-algebra $B \subset K$ with $h \in G(B(i))$. Since B is real and k is real closed, there exists, by the real Nullstellensatz, a k-linear homomorphism $\phi : B \to k$ with $\phi(1) = 1$. Applying ϕ to the identity $c(\sigma) = h^{-1}\sigma(h)$ one obtains $c(\sigma) = \phi(h)^{-1}\sigma(\phi(h))$. Thus c is a trivial 1-cocycle and there is an isomorphism $\omega_1 \to \omega_2$. Hence L_1 and L_2 are isomorphic as differential field extensions of K.

Remark. The natural map $H^1(\{1,\sigma\}, G(k(i))) \to H^1(\{1,\sigma\}, G(K(i)))$ is injective, by the above argument. $\square$

3 Comments and Examples

The proof of the unicity of a real Picard–Vessiot field uses almost exclusively properties of Tannakian categories. This implies that the proof remains valid for other types of equations, such as:

(a). linear partial differential equations, like $\frac{\partial}{\partial z_j} Y = A_j Y$ for $j = 1, \ldots, n$,

(b). linear ordinary difference equations, like $Y(z + 1) = AY(z)$,

(c). linear q-difference equations with $q \in \mathbb{R}^*$, like $Y(qz) = AY(z)$.

For case (a), the existence of a real Picard-Vessiot field has been proved in [CH]. The proof of the uniqueness result (Theorem 1.1) for real differential

fields with real closed field of constants can probably be rephrased for the case of differential modules over a formally p-adic differential field with a p-adically closed field of constants of the same rank.

Observations 3.1
Let K be a real closed differential field with field of constants k, M/K a differential module and $\omega :< M >_\otimes \to vect(k)$ a fibre functor. Let L be the Picard–Vessiot field corresponding to ω and G the group of the differential automorphisms of L/K. Let H be the differential Galois group of $K(i) \otimes M$ over $K(i)$. We recall that G is a form of H over the field $k(i)$. Using the identification $k(i) \times_k G = H$, one obtains on H and on $Aut(H)$ a structure of algebraic group over k. Let $\{1, \sigma\}$ be the Galois group of $k(i)/k$. Then $H^1(\{1, \sigma\}, Aut(H))$ has a natural bijection to the set of forms of H over k. Although the action of σ on $Aut(H)$ depends on G, this set does not depend on the choice of G.

Let $\eta :< M >_\otimes \to vect(k)$ be another fibre functor. Then η is mapped, according to Proposition 1.4, to an element in $\xi(\eta) \in H^1(\{1, \sigma\}, G(k(i)))$ (and this induces a bijection between η's and elements in this cohomology set). A 1-cocycle c for the group $\{1, \sigma\}$ has the form $c(1) = 1,\ c(\sigma) = a$ and a should satisfy $a \cdot \sigma(a) = 1$ (and is thus determined by a).

A 1-cocycle for $\xi(\eta)$ can be made as follows. The fibre functor η corresponds to a Picard–Vessiot field L_η. Both $L(i)$ and $L_\eta(i)$ are Picard–Vessiot fields for $K(i) \otimes M$ over $K(i)$. Thus there exists a $K(i)$-linear differential isomorphism $\phi : L(i) \to L_\eta(i)$. On the field $L(i)$ we write τ for the conjugation given by $\tau(i) = -i$ and τ is the identity on L. The similar conjugation on $L_\eta(i)$ is denoted by τ_η. Now $\tau_\eta \circ \phi \circ \tau : L(i) \to L_\eta(i)$ is another $K(i)$-linear differential isomorphism. A 1-cocycle c for $\xi(\eta)$ is now $c(\sigma) = \phi^{-1} \circ \tau_\eta \circ \phi \circ \tau$.

Let G_η denote the group of the K-linear differential automorphism of L_η. The group G_η is a form of G and produces an element in $H^1(\{1, \sigma\}, Aut(H))$ with $H = k(i) \times G$. We want to compute a 1-cocycle C for this element. Define the isomorphism $\psi : k(i) \times G \to k(i) \times G_\eta$ of algebraic groups over $k(i)$, by $\psi(g) = \phi \circ g \circ \phi^{-1}$. Define τ_G, the 'conjugation' on $k(i) \times G$, by the formula $\tau_G(g) = \tau \circ g \circ \tau$ for the elements $g \in G(k(i))$. Let τ_{G_η} be the similar conjugation on $k(i) \times G_\eta$. Now $\tau_{G_\eta} \circ \psi \circ \tau_G : k(i) \times G \to k(i) \times G_\eta$ is another isomorphism between the algebraic groups over $k(i)$. The 1-cocycle C is given by $C(\sigma) = \psi^{-1} \circ \tau_{G_\eta} \circ \psi \circ \tau_G$. One observes that $C(\sigma)(g) = c(\sigma) g c(\sigma)^{-1}$.

The map, which associates to $h \in G(k(i))$, the automorphism $g \mapsto hgh^{-1}$ of G, induces a map $H^1(\{1,\sigma\}, G(k(i))) \to H^1(\{1,\sigma\}, G/Z(G)(k(i))) \to H^1(\{1,\sigma\}, Aut(H))$, denoted by $\xi(\eta) \mapsto \tilde{\xi}(\eta)$. The forms corresponding to elements in the image of $H^1(\{1,\sigma\}, G/Z(G)(k(i))) \to H^1(\{1,\sigma\}, Aut(H))$ are called 'inner forms of G'. By §1, η induces a Picard–Vessiot field and a form $G(\eta)$ of H. Above we have verified (see [B] for a similar computation) that $G(\eta)$ is the inner form of G corresponding to the element $\tilde{\xi}(\eta)$. For the delicate theory of forms we refer to the informal manuscript [B] and the standard text [Sp]. □

Examples.

We continue with the notation and assumptions of Observations (3.1).

(1). Let $M/K, \omega, L, G$ be such that $G = \mathrm{SL}_{n,k}$. Since $H^1(\{1,\sigma\}, \mathrm{SL}_n(k(i))$ is trivial, L is the unique Picard–Vessiot field and is a real field (because a real Picard–Vessiot field exists).

The group SL_n has non trivial forms. For instance, $\mathrm{SU}(2)$ is an inner form of $\mathrm{SL}_{2,\mathbb{R}}$. There are examples, according to Proposition 3.2 below, of differential modules M/K having a real Picard–Vessiot field L with group of differential automorphisms of L/K equal to $\mathrm{SU}(2)$.

From [Sp], 12.3.7 and 12.3.9 one concludes that $H^1(\{1,\sigma\}, \mathrm{SU}(2)(\mathbb{C}))$ is trivial. Again L is the only Picard–Vessiot field.

(2). If G is the symplectic group $\mathrm{Sp}_{2n,k}$, then there are no forms and $H^1(\{1,\sigma\}, G(k(i)))$ is trivial. Therefore there is only one Picard–Vessiot field L and this is a real field.

(3). Consider a k-form G of $\mathrm{SO}(n)_k$ with odd $n \geq 3$. The center Z of G consists of the scalar matrices of order n, thus Z is the group $\mu_{n,k}$ of the nth roots of unity. Since n is odd, one has $Z(k) = \{1\}$. Further, again since n is odd, the automorphisms of $H = G_{k(i)}$ are interior and $Aut(H)(k(i)) = G/Z(k(i))$. We claim the following.

The natural map $H^1(\{1,\sigma\}, G(k(i))) \to H^1(\{1,\sigma\}, G/Z(k(i)))$ *is a bijection.*

Proof. A 1-cocycle c for $G/Z(k(i))$ is given by $c(1) = 1$ and $c(\sigma) = a \in G/Z(k(i))$ with $a\sigma(a) = 1$. Choose an $A \in G(k(i))$ which maps to a. Thus $A\sigma(A) \in Z(k(i))$ and A commutes with σA. Further $\sigma(A\sigma(A)) = \sigma(A)A = A\sigma(A)$ and thus $A\sigma(A) \in \mu_n(k) = \{1\}$. Therefore C defined by $C(1) = 1$, $C(\sigma) = A$ is a 1-cocycle for $G(k(i))$ and maps to c. Hence the map is surjective.

Consider for $j = 1, 2$ the 1-cocycle C_j for G given by $C_j(\sigma) = A_j$. Suppose that the images of C_j as 1-cocycles for $G/Z(k(i))$ are equivalent. Then there exists $B \in G(k(i))$ such that $B^{-1}A_1\sigma(B) = xA_2$ for some element $x \in Z(k(i))$. We may replace B by yB with $y \in Z(k(i))$. Then x is changed into $xy^{-1}\sigma(y)$. And the latter is equal to 1 for a suitable y. This proves the injectivity of the map. $\square$

We conclude from the above result that there exists a (unique up to isomorphism) fibre functor $\eta :< M >_\otimes \to vect(k)$ (or, equivalently, a Picard–Vessiot field) for every form of $H = SO(n)_{k(i)}$ over k. Moreover, only one of these fibre functors corresponds to a *real* Picard–Vessiot field.

Let $\omega :< M >_\otimes \to vect(k)$ denote the fibre functor corresponding to a *real* Picard–Vessiot field L_ω and G_ω the group of the differential automorphisms of L_ω/K. *We want to identify this form G_ω of $H := \mathrm{SO}(n)_{k(i)}$.*

Since the differential Galois group of $K(i) \otimes M$ is $\mathrm{SO}(n)_{k(i)}$, there exists an element $F \in sym^2(K(i) \otimes M^*)$ with $\partial F = 0$. Further F is unique up to multiplication by a scalar and F is a non degenerate bilinear symmetric form. The non trivial automorphism σ of $K(i)/K$ and of $k(i)/k$ acts in an obvious way on $K(i) \otimes M$ and on constructions by linear algebra of $K(i) \otimes M$. Now $\sigma(F)$ has the same properties as F and thus $\sigma(F) = cF$ for some $c \in K(i)$. After changing F into aF for a suitable $a \in K(i)$, we may suppose that $\sigma(F) = F$. Then F belongs to $sym^2(M^*)$ and is a non degenerate form of degree n over the field K. Further F is determined by its signature because K is real closed. Moreover KF is the unique 1-dimensional submodule of $sym^2(M^*)$. We claim the following:
G_ω is the special orthogonal group over k corresponding to a form f over k which has the same signature as F.

Let $V = \omega(M)$. The group G_ω is the special orthogonal group of some non degenerate bilinear symmetric form $f \in sym^2(V^*)$. Since L_ω is real, there exists a isomorphism $m : K \otimes_k \omega \to \rho$ of functors. Applying m to the modules M and $sym^2(M^*)$ one finds an isomorphism $m_1 : K \otimes_k V \to M$ of K-vector spaces which induces an isomorphism of K-vector spaces $m_2 : K \otimes_k sym^2(V^*) \to sym^2(M^*)$. The latter maps the subobject $K \otimes kf$ to KF by the uniqueness of KF. One concludes that the forms f and F have the same signature.

Proposition 3.2 *Suppose that K is real closed. Given is a connected semi-simple group H over $k(i)$ and a form G of H over k. Then there exists a differential module M over K and a real Picard–Vessiot field for M/K such that the group of the differential automorphisms of L/K is G.*

Proof. Let G be given as a subgroup of some $\mathrm{GL}_{n,k}$, defined by a radical ideal I. Then $k[G] = k[\{X_{k,l}\}_{k,l=1}^{n}, \frac{1}{\det}]/I$. The tangent space of G at $1 \in G$ can be identified with the k-linear derivations D of this algebra, commuting with the action of G. These derivations D have the form $(DX_{k,l}) = B \cdot (X_{k,l})$ for some matrix $B \in Lie(G)(k)$ (where $Lie(G) \subset \mathrm{Matr}(n,k)$ is the Lie algebra of G).

The same holds for $K[G] = K \otimes_k k[\{X_{k,l}\}_{k,l=1}^{n}, \frac{1}{\det}]/I$. Any K-linear derivation D on the algebra, commuting with the action of G, has the form $(DX_{k,l}) = A \cdot (X_{k,l})$ with $A \in Lie(G)(K)$. We choose A as general as possible.

The differential module M/K is defined by the matrix equation $y' = Ay$. It follows from [PS], Proposition 1.3.1 that the differential Galois group of $K(i) \otimes M$ is contained in $H = G_{k(i)}$. Now one has to choose A such that the differential Galois group (which is connected because $K(i)$ is algebraically closed) is not a proper subgroup of H. Since H is semi-simple, there exists a Chevalley module for H. Using this Chevalley module one can produce a general choice of A such that differential Galois group of $y' = Ay$ over $K(i)$ is in fact $G_{k(i)}$ (compare [PS], §11.7 for the details which remain valid in the present situation).

The usual way to produce a Picard–Vessiot ring for the equation $y' = Ay$ is to consider the differential algebra $R_0 := K[\{X_{k,l}\}_{k,l=1}^{n}, \frac{1}{\det}]$, with differentiation defined by $(X'_{k,l}) = A \cdot (X_{k,l})$, and to produce a maximal differential ideal in R_0. Since $A \in Lie(G)(K) \subset Lie(H)(K(i))$, the ideal $J \subset R_0[i]$, generated by I is a differential ideal. It is in fact a maximal differential ideal of $R_0[i]$, since the differential Galois group is precisely H. Then $J \cap R_0 = IR_0$ is a maximal differential ideal of R_0 and $K[G] = R = R_0/IR_0$ is a Picard–Vessiot ring for M over K. The field of fractions L of R is real because the G-torsor $Spec(K[G])$ is trivial. □

It seems that, imitating the proofs in [MS], one can show that Proposition 3.2 remains valid under the weaker conditions: K is a real differential field and a C_1-field and H is connected.

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